THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH3070 (Second Term, 2014–2015)
Introduction to Topology
Exercise 8 Compactness

Remarks

Many of these exercises are adopted from the textbooks (Davis or Munkres). You are suggested to work more from the textbooks or other relevant books.

1. Let $(X, \mathcal{T})$ be given the cofinite topology.
   (a) Show that X is compact.
   (b) Show that every subset $A \subset X$ is compact.
   (c) Do you think the same hold for co-countable topology?

2. A family $\mathcal{F}$ of closed sets satisfies *finite intersection property* if every intersection of finitely many sets in $\mathcal{F}$ is nonempty. Prove that the following is equivalent to compactness: every family $\mathcal{F}$ of closed sets satisfying the finite intersection property must have $\cap\mathcal{F}$ nonempty.

   A family $\mathcal{C}$ of sets (not necessarily open nor closed) satisfies *finite closure intersection property* if for each finite $\mathcal{A} \subset \mathcal{C}$, the intersection $\cap \left\{ \overline{A} : A \in \mathcal{A} \right\} \neq \emptyset$. Show that compactness is equivalent to: every family $\mathcal{C}$ satisfying the finite closure intersection property must have $\cap \left\{ \overline{C} : C \in \mathcal{C} \right\} \neq \emptyset$.

3. Let $\mathcal{B}$ be a base for $\mathcal{T}$. Assume that every open cover $\mathcal{C} \subset \mathcal{B}$ for X has a finite subcover. Prove that X is compact. *Remark*. The converse is trivially true.

   Remark. The same question concerning subbase is considerably harder.

4. Show that if a space $(X, \mathcal{T})$ is compact and discrete then X is finite.

5. Use the indiscrete topology to create an example of a compact space X with a compact subset A which is not closed in X.

6. Let X be compact and $F_n \subset X$ be nonempty closed sets such that $F_{n+1} \subset F_n$ for each $n \in \mathbb{N}$. Show that $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$.

7. Recall that a set S in a metric space (Y, d) is *bounded* iff $S \subset B(y_0, R)$ for some $y_0 \in Y$ and $R > 0$. Let (X, d) be a metric space. Prove that if $K \subset X$ is compact, it is closed and bounded.

   Do you think the converse is true?

8. Prove that if $(X, \mathcal{T})$ is compact and $f\colon (X, \mathcal{T}) \to (Y, d)$ is continuous, then the image $f(X)$ is bounded.

9. Let K_α be compact subsets in a topological space $(X, \mathcal{T})$. Prove that a finite union of K_α's is compact and, if X is Hausdorff, an arbitrary intersection of K_α's is compact.

   Think about what happens to infinite union of compact sets.

10. Let $C(X) = \{\, f\colon X \to \mathbb{R} \mid f \text{ is continuous} \,\}$. Prove that if X is compact, then

$$d(f, g) = \sup \{\, |f(x) - g(x)| \;:\; x \in X \,\}$$

   defines a metric on $C(X)$.